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Section A.—Mathematical and Physical Sciences.

On the Longitudinal Strength of Cylinders closed by Screw Plugs.

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When designing the breech end of a gun, or the screwed end of a hydraulic cylinder, it has been customary to assume that the longitudinal stress due to the powder pressure in a gun, or to the water pressure in a cylinder, is uniformly distributed over the cross-section of the gun tube or of the cylinder, as the case may be.

This, however, is seldom the case, because in practice the stress can rarely be applied uniformly over the section, and, therefore, almost always acts either on the inner or on the outer circumferential surface. The stress on the material is consequently greatest on the surface where the force is applied; thence, owing to the fibres stretching and sliding longitudinally over each other, the stress is partly transmitted throughout the thickness of the cylinder.

We shall show that in a cylinder there is, on the screwed end, a bending moment exerted on the material, in addition to the ordinary uniform longitudinal stress.

To reduce the investigation to its simplest case: Suppose that we have a cylinder closed by means of a screwed plug and subjected to an internal pressure exerted within the cylinder. We desire to ascertain the greatest stress on the screwed part of the cylinder.

Let the screw thread be of V form, of which the thrust side makes an angle α with the axis of the cylinder.

Let a' be the total area (square inches) of the thrust surface of the screw thread, l the length in inches of the screwed portion, p_0 the internal pressure (tons per square inch), r_0 the internal radius of the cylinder to which the pressure is confined, r_1 the radius of the screwed part at the root of the thread, r_1' the mean radius of the screw thread, r_2 the external radius of the cylinder, λ the angle of friction of the screwed surfaces.

The total longitudinal pressure, in tons, on the screw plug is

$$\pi p_0 r_0^2.$$

Then if the depth EA of the screw thread ABD (fig. 1) is taken instead of the inclined surface AB, we can put

$$a' \sin \alpha = 2\pi r_1' a,$$

where $2\pi r_1' a$ is the total helical surface formed by the line EA.

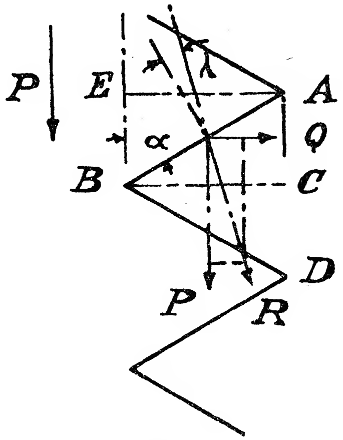

Fig. 1.

The pressure P acts on the thrust side AB of the thread, giving rise to a reaction R and an outward thrust Q.

If p_1 is the pressure per square inch corresponding to P and p_2 and p_3 the pressures per square inch corresponding to R and Q respectively, then we have

$$p_1 \times \mathrm{BC} = \mathrm{P}, \qquad p_2 \times \mathrm{AB} = \mathrm{R}, \qquad p_3 \times \mathrm{AC} = \mathrm{Q}.$$

Now $$\mathrm{P} = \mathrm{R} \sin(\alpha+\lambda) \quad \text{and} \quad \mathrm{Q} = \mathrm{P} \cot(\alpha+\lambda),$$

or $$p_3 \times \mathrm{AC} = p_1 \times \mathrm{BC} \cot(\alpha+\lambda),$$

$$p_3 = p_1 \mathrm{BC}/\mathrm{AC} \, . \, \cot(\alpha+\lambda)$$

$$= p_1 \tan \alpha \cot(\alpha+\lambda),$$

and the curious result is obtained that

$$p_3 = p_1 \quad \text{if} \quad \lambda = 0.$$

As p_3 acts on AC or on only half of the width of the root of the thread, we may, by halving p_3, consider it to act on the whole depth. Thus the thrust outwards becomes

$$\tfrac{1}{2}p_1 \tan\alpha \cot(\alpha+\lambda) \text{ per square inch on the inner surface of the cylinder of radius } r_1,$$

but as
$$p_1 = \pi p_0 r_0^2 / 2\pi r_1' a = p_0 r_0^2 / 2 r_1' a$$
the outward thrust is:
$$p_0 r_0^2 / 4 r_1' a \,.\, \tan\alpha \cot(\alpha+\lambda). \tag{1}$$

This will manifest itself as a bursting stress on the threaded part AB, fig. 2, of the cylinder.

If the screw thread is of the buttress type α becomes $\pi/2$, and as the

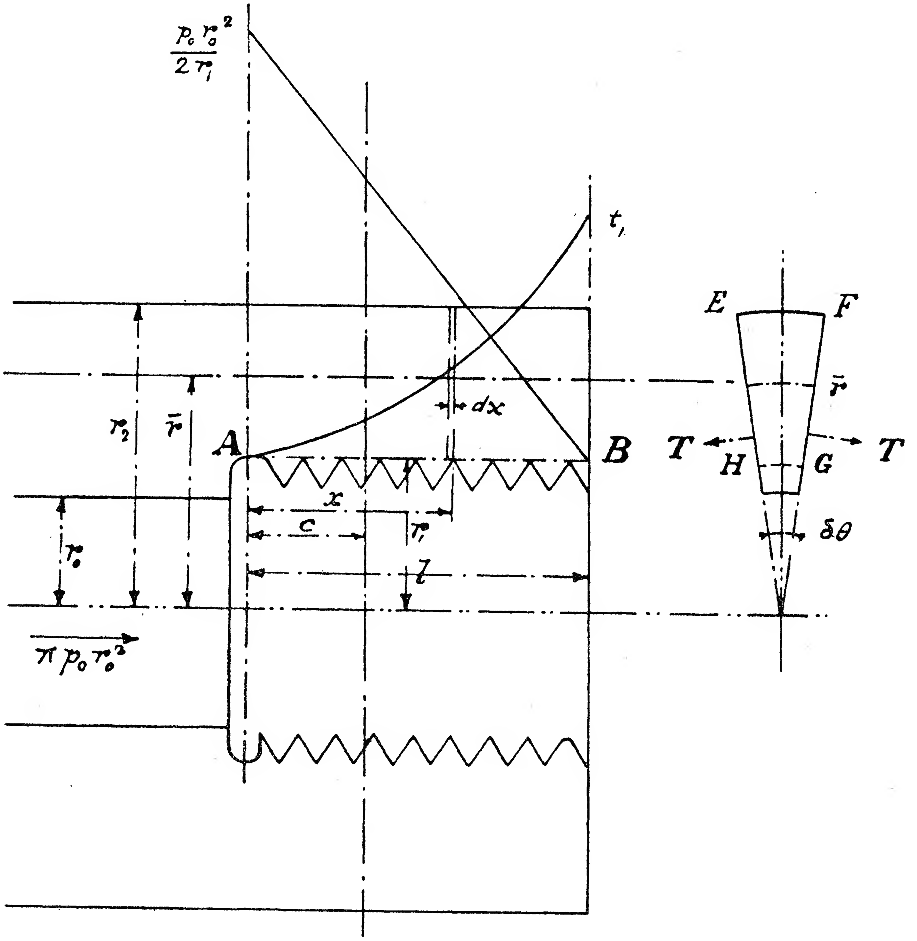

Fig. 2.

longitudinal force is normal to its surface there is no tendency to slide: consequently the outward stress vanishes.

The total pressure $\pi p_0 r_0^2$ on the screw plug is taken on the wall of the cylinder at the root of the screw thread.

Here, the surface exposed to this longitudinal force is $2\pi r_1 l$, and there will, consequently, be an uniformly distributed but cumulative longitudinal stress of

$$p_0 r_0^2/2r_1 l \text{ tons per square inch.}$$

At B (fig. 2) this stress is 0, but at A it has an intensity

$$p_0 r_0^2/2r_1 \text{ tons per square inch.}$$

In order to examine the stress on the cylinder divide it longitudinally into an infinite number of elementary segments EFGH, and consider the forces acting on one segment embraced by two radial planes at an angular distance $\delta\theta$ apart.

Then the width of this segmental beam at radius r_1 is $r_1\delta\theta$: and as the longitudinal force is uniformly increasing over the screwed internal surface, it is zero at B (fig. 2) and has its greatest value, $p_0 r_0^2/2r_1$ tons per square inch, at A. The total longitudinal force on the width $r_1\delta\theta$ of the beam at any cross-section distant c from A is therefore

$$\frac{l-c}{l}\frac{p_0 r_0^2}{2r_1} r_1\delta\theta.$$

This force tends to stretch the material longitudinally; it will also tend to bend the beam about its neutral surface, and as $\delta\theta$ can be made indefinitely small, the neutral surface may, for each elementary beam, be considered a plane surface.

If $\bar{r}$ denotes the radius of this neutral surface the force

$$\frac{l-c}{l}\frac{p_0 r_0^2}{2r_1} r_1\delta\theta$$

acts at a distance $\bar{r}-r_1$ from it.

The tension, or positive, stress ${}_r p_c$ on the material of the beam, at any radius r of the section is, therefore, the uniformly distributed stress on the section due to the load *plus* the stress due to the bending moment on the section. Thus

$$ {}_r p_c = \frac{p_0 r_0^2}{r_2^2 - r_1^2}\frac{l-c}{l} + \frac{\mathrm{M}_c(\bar{r}-r)}{\mathrm{I}}, \qquad (2)$$

where M_c is the resultant bending moment on the beam about the neutral axis of the section under consideration, and I the moment of inertia of the section.

Note, however, that when r is greater than $\bar{r}$, the last term becomes negative, and a compressive stress may result on the outer circumferential fibres.

The resultant bending moment,

$$\mathrm{M}_c = \frac{p_0 r_0^2}{2 r_1} r_1 \delta\theta \frac{l-c}{l} (\bar{r} - r_2) +$$

$$\frac{p_0 r_0^2}{4 r_1' a} \tan\alpha \cot(\alpha+\lambda)\, r_1' \delta\theta \int_c^l (x-c)\, dx - \delta\theta \int_c^l \mathrm{T}\,(x-c)\, dx. \qquad (3)$$

The first term on the right is the moment of the longitudinal force, acting at radius r_1, about the neutral axis at radius $\bar{r}$; the second term is the sum of the moments of the bursting stress due to the thrust component of the screw threads. This thrust is uniform throughout the length l and acts about the point D (fig. 2); its magnitude, on the mean width $r_1'\delta\theta$ of the screw thread and for a length dx of the beam, is

$$p_0 r_0^2 / 4 r_1' a \;.\; \tan\alpha \cot(\alpha+\lambda)\, r_1' \delta\theta\, dx,$$

for any distance x from A the leverage about D is $(x-c)$, consequently the moment is

$$p_0 r_0^2 / 4 r_1' a \;.\; \tan\alpha \cot(\alpha+\lambda)\, r_1' \delta\theta\, (x-e)\, dx:$$

and the sum of the moments from l to c is that shown in the second term.

The third term, acting in the contrary direction, is the sum of the moments of the hoop tension resolved parallel to the central plane of the segmental beam; it is greatest at B and vanishes at A, where the deflection of the beam outwards is least. The total tension effort, for a thin strip dx across one of the bounding radial planes, is T, suppose, and we may assume that the distribution of the tension through the thickness $r_2 - r_1$ of the cylinder follows the usual laws for thick tubes.

This is, however, only strictly correct when the longitudinal stress is uniform throughout the section. T is, therefore, the sum of the circumferential tensions through the thickness of the cylinder.

It should be noted that c can be replaced by x, and that when $c = 0$, p_c has its greatest value; and that when $c = l$, $p_c \doteq 0$, *i.e.* at the extreme end of the cylinder.

Again, from elementary principles, we have for the deflection u of the beam

$$\mathrm{EI} \frac{d^2 u}{dx^2} = \mathrm{M}_c.$$

Therefore from equation (2), putting $r = r_1$ in the last term for the greatest stress,

$$\mathrm{EI} \frac{d^2 u}{dx^2} = \frac{\mathrm{I}}{\bar{r} - r_1} \left(p_c - \frac{p_0 r_0^2}{r_2^2 - r_1^2} \frac{l-c}{l} \right), \qquad (4)$$

changing c into x and integrating

$$\mathrm{E}\,\frac{du}{dx} = \frac{1}{\bar{r}-r_1}\left(p_x x - \frac{p_0 r_0^2}{r_2^2-r_1^2}\,\frac{lx-\frac{1}{2}x^2}{l}\right).$$

As $du/dx = 0$ when $x = 0$, there is no constant.

Integrating again

$$\mathrm{E}u = \frac{1}{\bar{r}-r_1}\left(\tfrac{1}{2}p_x x^2 - \frac{p_0 r_0^2}{r_2^2-r_1^2}\,\frac{\frac{1}{2}lx^2-\frac{1}{6}x^3}{l}\right). \tag{5}$$

Here at the neutral surface the deflection $u = 0$ when $x = 0$, and, therefore, there is no constant; also u is greatest when $x = l$, in which case $p_x = 0$.

For the greatest deflection put $x = l$, then equation 5 becomes

$$\mathrm{E}u = -\frac{1}{3}\,\frac{1}{\bar{r}-r_1}\,\frac{p_0 r_0^2}{r_2^2-r_1^2}\,l^2, \tag{6}$$

the negative sign indicating the direction of the deflection.

From the usual formulas for thick cylinders

$$\mathrm{E}\;u/\bar{r} = t - \sigma\,(p + {}_{\bar{r}}p_x) \tag{7}$$

where t is the circumferential tension at radius $\bar{r}$, p the radial pressure (a negative value), and ${}_{\bar{r}}p_x$ the longitudinal stress, both at radius $\bar{r}$.

When $x = l$, $p = 0$, a constant, so that the ordinary gun formulas hold at this point.

From the ordinary gun formulas for the interior layers of metal in a thick cylinder, viz.,

$$p = p_1\frac{r_1^2}{r^2}\,\frac{r_2^2-r^2}{r_2^2-r_1^2} + p_2\frac{r_2^2}{r^2}\,\frac{r^2-r_1^2}{r_2^2-r_1^2}$$

$$t = p_1\frac{r_1^2}{r^2}\,\frac{r_2^2+r^2}{r_2^2-r_1^2} - p_2\frac{r_2^2}{r^2}\,\frac{r^2+r_1^2}{r_2^2-r_1^2}$$

and

$$p_1 = \frac{r_2^2-r_1^2}{r_2^2+r_1^2}(t_1+p_2)+p_2.$$

In our case as we are considering a single cylinder only, $p_2 = 0$ as there is no external pressure.

If, however, an outer strengthening cylinder is shrunk on to the main cylinder

$$p_2 = \frac{r_3^2-r_2^2}{r_3^2+r_2^2}\,t_2 \quad \text{and} \quad \mathrm{T} = p_1 r_1 - p_2 r_2.$$

Consequently in (7)

$$t = t_1\frac{r_1^2}{r_2^2+r_1^2}(1+r_2^2/\bar{r}^2),$$

$$p = t_1\frac{r_1^2}{r_2^2+r_1^2}(r_2^2/\bar{r}^2-1),$$

where t_1 is the circumferential tension of the inner layer at the extreme end B of the cylinder. From equations (6) and (7) we obtain—by putting Poisson's ratio $= \frac{1}{4}$ and inserting the values just found for t and p

$$t_1\bar{r}\,\frac{r_1^2}{r_2^2+r_1^2}\left(\frac{3}{4}+\frac{5}{4}\frac{r_2}{\bar{r}^2}\right) = \frac{1}{3}\frac{1}{\bar{r}-r_1}\frac{1}{r_2^2-r_1^2}p_0r_0^2l^2,$$

so that

$$t_1 = \frac{4}{3}\frac{r_2^2+r_1^2}{r_2^2-r_1^2}\frac{p_0r_0^2}{\bar{r}r_1^2(\bar{r}-r_1)}\frac{l^2}{(3+5r_2^2/\bar{r}^2)}\text{ tons/inch}^2. \qquad (8)$$

In order to find the value of p_x (when $c = x$) in equations (2) and (3), some law must be assumed for the variation of t_x in the expression

$$\int_c^l \mathrm{T}\,(x-c)\,dx.$$

If the ordinary formulas for thick cylinders be adopted as the most convenient, although not perhaps absolutely exact, then, since there is no external pressure,

$$\mathrm{T} = p_1r_1 \quad \text{and } p_1 = \frac{r_2^2-r_1^2}{r_2^2+r_1^2}t_x,$$

where p_1 is the internal pressure which, acting on a diameter $2r_1$, would be required, at each section considered, to produce the total tension 2T.

From the form of equations (5) and (8) it is reasonable to assume that the inner circumferential tension t_x increases, proportionally to x^2, from zero at A to t_1 at B. If this be assumed, then at any section distant x from A

$$t_x = \frac{x^2}{l^2}t_1.$$

And

$$\int_c^l \mathrm{T}\,(x-c)\,dx = \frac{r_1\,t_1}{l^2}\frac{r_2^2-r_1^2}{r_2^2+r_1^2}\int_c^l x^2\,(x-c)\,dx = \frac{r_1\,t_1}{l^2}\frac{r_2^2-r_1^2}{r_2^2+r_1^2}\left(\tfrac{1}{4}x^4-\tfrac{1}{3}x^3c\right)_c^l$$

when $x = 0$, T or $t_x = 0$, and there is therefore no constant.

Inserting the limits

$$\int_c^l \mathrm{T}\,(x-c)\,dx = \frac{r_1\,t_1}{l^2}\frac{r_2^2-r_1^2}{r_2^2+r_1^2}\left[l^3\left(\tfrac{1}{4}l-\tfrac{1}{3}c\right)+\tfrac{1}{12}c^4\right]. \qquad (9)$$

At A, where $c = 0$, the expression assumes its greatest value, viz.:—

$$\tfrac{1}{4}r_1t_1\frac{r_2^2-r_1^2}{r_2^2+r_1^2}l^2,$$

and when $c = l$ it becomes zero.

The value of ${}_rp_c$ can now be found at any point from equations (2) and (3) by replacing the various terms by their proper values; generally it is only necessary to know the greatest stress the material has to support. Thus when $c = 0$ and $r = r_1$ the greatest stress ${}_{r_1}p$ is found, *i.e.*

$$
{}_{r_1}p = \frac{p_0 r_0^2}{r_2^2 - r_1^2} + \frac{\bar{r} - r_1}{\mathrm{I}} \left(\frac{p_0 r_0^2}{2 r_1} r_1 \delta\theta \, (\bar{r} - r_1) \right.
$$

$$
\left. + \frac{p_0 r_0^2}{8 r_1' a} \tan\alpha \cot(\alpha + \lambda) \, r_1' \, \delta\theta l^2 - \tfrac{1}{4} r_1 t_1 \frac{r_2^2 - r_1^2}{r_2^2 + r_1^2} \delta\theta l^2 \right).
$$

When $\delta\theta$ is very small it will be found that

$$
\mathrm{I} = \tfrac{1}{4} \delta\theta \, (r_2^2 - r_1^2) \, (r_2^2 + r_1^2 - 2\bar{r}^2) \qquad (10)
$$

and replacing t_1 by its value in equation (8) we finally obtain

$$
{}_{r_1}p = \frac{p_0 r_0^2}{r_2^2 - r_1^2} \left[1 + \frac{1}{r_2^2 + r_1^2 - 2\bar{r}^2} \left(2\,(\bar{r} - r_1)^2 \right. \right.
$$

$$
\left. \left. + (\bar{r} - r_1) \frac{\tan\alpha \cot(\alpha + \lambda)}{2a} l^2 - \frac{4}{3} \frac{l^4}{\bar{r}\, r_1 (3 + 5 r_2/\bar{r}^2)} \right) \right] \text{ tons/inch}^2. \qquad (11)
$$

It follows that the length l should be so proportioned that

$$
2|(\bar{r} - r_1)^2 + (\bar{r} - r_1) \frac{\tan\alpha \cot(\alpha + \lambda)}{2a} l^2 - \frac{4}{3} \frac{l^4}{\bar{r} r_1 (3 + 5 r_2^2/r^2)} = 0. \qquad (12)
$$

The expression $\tan\alpha \cot(\alpha + \lambda)$ equals 1 when $\lambda = 0$, but for all ordinary values of λ up to 6° or 7° it is a little less than 1; if, therefore, we suppose it to be 1 a slightly larger value for the bursting moment will be obtained.

The formula will then be simplified to

$$
{}_{r_1}p = \frac{p_0 r_0^2}{r_2^2 - r_1^2} \left[1 + \frac{1}{r_2^2 + r_1^2 - 2\bar{r}^2} \left(2\,(\bar{r} - r_1)^2 + \frac{\bar{r} - r_1}{2a} l^2 - \frac{4}{3} \frac{l^4}{\bar{r} r_1 (3 + 5 r_2^2/\bar{r}^2)} \right) \right]
$$

$$
\text{tons/inch}^2. \qquad (13)
$$